The Wright Function: Its Properties, Applications, and Numerical Evaluation

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In this paper, some important elements of the theory of the Wright function ϕ defined by

$$\phi(\rho, \beta; z) := \sum_{k=0}^{\infty} \frac{z^k}{k!\Gamma(\beta + \rho k)}, \ \rho > -1, \ \beta \in \mathbb{R}, \ z \in \mathbb{C},$$

Γ being the well-known Euler Gamma-function are first discussed. In particular, its representations through the other special functions of the hypergeometric type, different integral representations, and its asymptotical behavior are considered.

The Wright function – along with the Mittag–Leffler function – plays a prominent role in the theory of the partial differential equations of the fractional order that are actively used nowadays for modeling of many phenomena including, e.g., the anomalous diffusion processes or in the theory of the complex systems. This function appears there both as a Green function in the initial-value problems for the model linear equations with the constant coefficients and as a special solution invariant under the groups of the scaling transformations of the fractional differential equations. In this paper, both of these applications are shortly introduced.

Whereas the analytical theory of the Wright function is already more or less well developed, its numerical evaluation is still an area of the active research. In this paper, the numerical evaluation of the Wright function is discussed with a focus on the case of the real axis that is very important for applications. In particular, several approaches are presented including the method of series summation, integral representations, and asymptotical expansions. In different parts of the complex plane different numerical techniques are employed. In each case, the estimates for accuracy of the computations are provided. An implementation of the algorithms in the pseudo-code is included, too. The presentation is accompanied by a number of plots and pictures obtained with the algorithms discussed in this paper.

The ideas and techniques employed in this paper can be used with some small modifications for the numerical evaluation of other functions of the hypergeometric type.

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