The Multi-index Mittag-Leffler Functions and Their Applications for Solving Fractional Order Problems in Applied Analysis

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During the last few decades differential equations and systems of fractional order (that is, of arbitrary one, not necessarily integer), begun to play an important role in modelling of various phenomena of physical, engineering, biological and biomedical, chemical, earth, economics, etc. nature. The so-called *"Special Functions of Fractional Calculus"* (SF of FC) provide an important tool of FC: In particular, they are often used to represent the solutions of the fractional differential equations in explicit form. Among the most prominent representatives of the SF of FC are the Wright generalized hypergeometric function ${}_p\Psi_q$, the more general Fox H-function, and the Inayat-Hussain $\bar{H}$-function. The classical special functions, among them the orthogonal polynomials and the ${}_pF_q$-hypergeometric functions, can be represented as particular cases of the SF of FC.

In this survey talk, an overview of some properties and applications of an important class of SF of FC, introduced for the first time in our works, is given. For an integer $m > 1$ and the arbitrary real (or complex) indices $\rho_1, \dots, \rho_m > 0$ and $\mu_1, \dots, \mu_m$, the *multi-index (vector-index) Mittag-Leffler functions* are defined by:

$$E_{(1/\rho_i),(\mu_i)}(z) := E^{(m)}_{(1/\rho_i),(\mu_i)}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\mu_1 + \frac{k}{\rho_1}) \dots \Gamma(\mu_m + \frac{k}{\rho_m})} = {}_1\Psi_m \left[\begin{matrix} (1,1) \\ (\mu_1, 1/\rho_i)_1^m \end{matrix} ; z \right] . \qquad (1)$$

The studies on these SF of FC were initialized in the monographs [1] and [2] and continued in a series of our papers till nowadays, some of them in collaboration with other colleagues from the family of the FC Analysts.

In the simplest case $m = 1$, the function (1) is reduced to the classical Mittag-Leffler (ML) function that is widely recognized nowadays as the Queen-function of FC. Other important particular cases of (1) are: the Wright function, the Bessel-Maitland, the Wright-Lommel, the Struve, the Lommel and the Airy functions, the Rabotnov and the Mainardi functions, the Lorenzo-Hartley R-function, the Dzhrbashjan function (ML function with 4 parameters, or function (1) for $m = 2$), the Delerue hyper-Bessel functions, etc. A lot of results connected with these and other special cases of (1) have been published recently in the journal [3].

In the talk, we survey some results on the analytical properties of the multi-index ML functions and their applications including a variety of their relations to the operational calculus and to the integral transforms, their asymptotics, distribution of their zeros, and some new integral and differential representations. The method used to derive many of these relations is based on the H-functions and the Generalized Fractional Calculus (GFC)-techniques. The functions (1) can be interpreted as the generating functions for a class of the Gelfond-Leontiev operators of the generalized integration and differentiation that are particular cases of the so-called multiple Erdélyi-Kober (EK) operators. Moreover, they are related to a Laplace type integral transform called the multi-index Borel-Dzhrbashjan transform, too. It can be shown that all SF of FC (in the sense of ${}_p\Psi_q$) can be represented in the form of the GFC operators acting on the three basic elementary functions and this fact suggests an unified approach to their classification into three classes. In particular, the multi-index ML functions (1) fall into the class of the trigonometric and the Bessel-type functions and can thus be seen as "fractional indices" analogues of the Delerue hyper-Bessel functions serving as solutions to differential equations of fractional multi-order $(1/\rho_1, 1/\rho_2, \ldots, 1/\rho_m)$.

Along with the analytical results for the functions (1), some examples of their applications in solving IVPs for the fractional differentiation equations with the multiple EK operators and in determining of the scale-invariant solutions of some partial differential equations of fractional order including the fractional diffusion-wave equation are presented.

Among the *open problems* we would like to attract the attention of the other researchers to, are the developments of reliable algorithms and computational procedures for evaluation of the functions (1) and other SF of FC. Some first results in this direction have been already obtained for the ML and the Wright functions, but the general problem is still waiting for its solution.

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References

[1] S.B. Yakubovich and Yu.F. Luchko, *The Hypergeometric Approach to Integral Transforms and Convolutions*, Kluwer, 1994.
[2] V. Kiryakova *Generalized Fractional Calculus and Applications*, Longman – J. Wiley, 1994.
[3] *Fractional Calculus and Applied Analysis* (FCAA, see http://www.math.bas.bg/~fcaa).

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